SOLUTION OF PROBLEMS ABOUT THE UNLOADING WAVE IN AN ELASTIC-PLASTIC MEDIUM SUBJECTED TO A PRANDTL SCHEME

A. V. Chernov

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Plane wave propagation in a semi-infinite rod of constant cross-section and in a half-space with a plane boundary filled with a continuous material, due to the action of a stress pulse applied to the endface of a rod along its axis or normally to the surface of the half-space, is described by a system of partial differential equations

$$\rho_0\frac{\partial u}{\partial t}+\frac{\partial \sigma}{\partial h}=0,\quad \frac{\partial u}{\partial h}+\frac{\partial \varepsilon}{\partial t}=0, \tag{1}$$

where ρ_0 is the initial density of the material, u is the mass flow rate of elementary layers of material arranged perpendicularly to the direction of wave propagation; t is the time, h is the Lagrange coordinate marking the initial position of the layer, σ is the stress acting in sections parallel to the wave front along the direction of its propagation in the case of the half-space; in the case of the rod, this is a conditional stress, i.e., a force acting in a certain section of the rod, referred to the initial cross-sectional area (considered positive under compression), $\varepsilon = -\partial W/\partial h$ is the deformation, and W is the displacement of the elementary layer.

The behavior of the separate elastic-plastic materials in the case of a uniaxial stress state realizable in the compression-tension of thin rods, or the case of uniaxial strain that is characteristic for the loading of materials by plane shocks, can be described approximately by a Prandtl scheme in some range of strains [1] (Fig. 1).

Loading a material from an initial undeformed state to stresses less than the yield point σ_s occurs purely elastically. The relation of the stress to the strain is subject to the linear relationship $\sigma = \rho_0 c_0^2 \varepsilon$, where c_0 is the velocity of longitudinal elastic wave propagation. In the case of the rod $\rho_0 c_0^2 = E$ (Young's modulus), and in the case of the half-space $\rho_0 c_0^2 = \lambda + 2\mu$ (λ and μ are the Lame elastic constants). Loading on the plastic section AB above the yield point is subject to the relationship $\sigma = \sigma_s + \rho_0 c^2(\varepsilon - \varepsilon_s)$, where $\rho_0 c^2$ is the strengthening modulus, and ε_s is the strain corresponding to the yield σ_s. Moreover, it is assumed in this model that unloading of the element of material that has achieved the state B is accomplished along the line BC parallel to the elastic section OA until a reverse plastic flow starts at the point C along the line CD parallel to AB.

Let us examine the wave process in an elastic-plastic material subject to the Prandtl scheme originating because of the action of the stress pulse shown at the upper left in Fig. 2, on the boundary. The initial stage of the pulse is smooth loading of the material on the boundary from zero to a maximum stress σ_m, the second stage is characterized by smooth unloading from σ_m to zero.

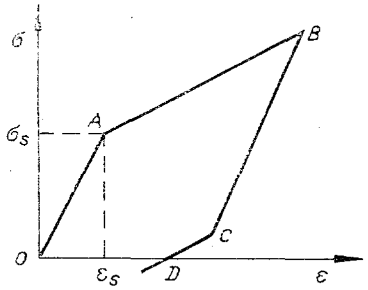

Fig. 1

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The qualitative wave pattern of the process in a plane (Fig. 2, upper right) was considered in detail in [1], where it was shown that the flow in this plane separates into a number of elastic and plastic domains separated by solid lines in the figures. An elastic predecessor in which the stress grows from zero to the line $h = c_0(t + T)$ is propagated in domain I, where $T > 0$ is the duration of the load rise, to σ_s on the line $h = c_0(t - t_A)$. The stress is constant and equal to σ_s in domain II. Domains III and IV are a plastic loading wave and an elastic unloading wave, respectively. They are separated by the line OM called the unloading wave in [1], on which continuity conditions for the stress, velocity, and strain are satisfied.

Taking account of the Prandtl σ–ε-diagram written in the characteristic form [2], the system (1) is easily integrated and results in the following relations:

a) In the elastic domains I and IV

$$\sigma + \rho_0 c_0 u = \text{const}; \tag{2}$$

is satisfied along the lines $h = c_0 t + \text{const}$, while

$$\sigma - \rho_0 c_0 u = \text{const}; \tag{2'}$$

is satisfied along the lines $h = -c_0 t + \text{const}$.

b) In the plastic domain III the relationships

$$\sigma \pm \rho_0 c u = \text{const}$$

are satisfied, respectively on the rectilinear C^+ and C^--characteristics $h = \pm\, ct + \text{const}$.

Let $f(t)$ denote the stress on the rising part of the boundary pulse for $t \leqslant 0$, and let $g(t)$ be the stress on the pulse drop for $t > 0$.

The solution is known in domains I, II and III:

$$\sigma(h,\ t) = \rho_0 c_0 u(h,\ t) = f(t - h/c_0) \qquad \text{(domain 1)},$$
$$\sigma(h,\ t) = \rho_0 c_0 u(h,\ t) = \sigma_s \qquad \text{(domain II)},$$
$$\sigma(h, t) = \sigma_s \frac{c_0 - c}{c_0} + \rho_0 c u(h, t) = f(t - h/c) \qquad \text{(domain III)}.$$

The main problem is to determine the form of the unloading curve OM and the flow in domain IV. Let $w(t)$ denote the velocity of the material on the boundary, and $H(t)$ the equation of the curve OM. Let us write the characteristic relationships operating along the C^+ characteristic BC and the C^- characteristic CD (see Fig. 2):

$$\begin{gathered} g\,[t - H(t)/c_0] + \rho_0 c_0 w\,[t - H(t)/c_0] = \frac{c_0 + c}{c} f\,[t - H(t)/c] - \frac{c_0 - c}{c}\sigma_s, \\ g\,[t + H(t)/c_0] - \rho_0 c_0 w\,[t + H(t)/c_0] = \frac{c_0 - c}{c}\sigma_s - \frac{c_0 - c}{c} f\,[t - H(t)/c]. \end{gathered} \tag{3}$$

The solution of the obtained system of functional equations (3) that determine the unknown functions w and H is quite complex, hence, we use the inverse method of solving the problem of an unloading wave [1] by assuming that the unloading wave is represented by a straight line, i.e., $H(t) = at$. This assumption permits obtaining an explicit analytic solution of the problem for certain simple but sufficiently interesting forms of the boundary stress pulse. Substituting $H(t) = at$ into the first equation of the system (3), and replacing $(c_0 - a)t/c_0$ by z, we obtain the relationship

$$g(z) + \rho_0 c_0 w(z) = \frac{c_0 + c}{c} f\left(\frac{c_0}{c}\,\frac{c - a}{c_0 - a} z\right) - \frac{c_0 - c}{c}\sigma_s.$$

The second equation yields the relation

$$g(z) - \rho_0 c_0 w(z) = \frac{c_0 - c}{c}\sigma_s - \frac{c_0 - c}{c} f\left(\frac{c_0}{c}\,\frac{c - a}{c_0 + a} z\right).$$

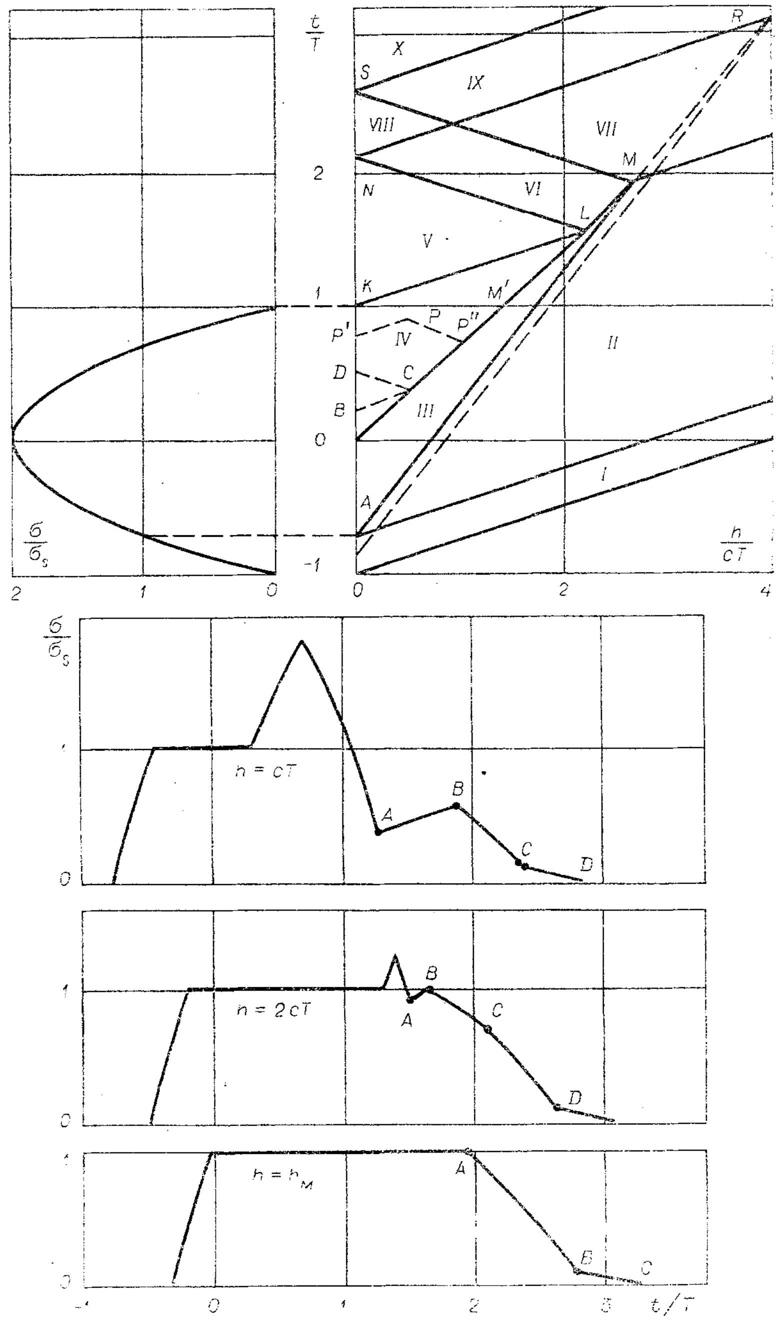

Fig. 2

The given form of the loading part of the pulse f(z) determines the stress g(z) and the mass flow rate w(z) uniquely on the dropping part of the pulse for t > 0:

$$g(z)=\frac{c_0+c}{2c}f\left(\frac{c_0}{c}\,\frac{c-a}{c_0-a}z\right)-\frac{c_0-c}{2c}f\left(\frac{c_0}{c}\,\frac{c-a}{c_0+a}z\right),$$

$$w(z)=\frac{1}{2\rho_0c_0}\left[\frac{c_0+c}{c}f\left(\frac{c_0}{c}\,\frac{c-a}{c_0-a}z\right)+\frac{c_0-c}{c}f\left(\frac{c_0}{c}\,\frac{c-a}{c_0+a}z\right)-2\frac{c_0-c}{c}\sigma_s\right]. \tag{4}$$

Let us assume that the function f(t) has the form

$$f(t)=\sigma_m[1-(-t/T)^{\alpha}], \tag{5}$$

for t $\leqslant 0$, where α is greater than zero. Then for t $\geqslant 0$ we have from (4) and (5)

$$\sigma(0, t) = g(t) = \sigma_m [1 - K(t/T)^\alpha], \tag{6}$$

where

$$K = \frac{1}{2c}\left[\frac{c_0(a-c)}{c}\right]^\alpha \left[\frac{c_0+c}{(c_0-a)^\alpha} - \frac{c_0-c}{(c_0+a)^\alpha}\right]. \tag{7}$$

The value of a varies within the range $c \leqslant a \leqslant c_0$, which is the necessary condition for the existence of an unloading wave [1, 3]. In this range (7) defines K as a function of a that grows monotonically from zero at $a = c$ to $+\infty$ as $a \to c_0$. Let us note that for $\alpha = 1$ and 2 and K = 1 formula (7) results in dependences of a on K to the accuracy of a notation in agreement with the formulas for the initial velocity of the unloading wave OM obtained earlier and presented in [1].

We obtain for the velocity of the material on the boundary for t $\geqslant 0$

$$u(0, t) = w(t) = u_m - \frac{\sigma_m}{\rho_0 c_0} L(t/T)^\alpha, \tag{8}$$

where

$$L = \frac{1}{2c}\left[\frac{c_0(a-c)}{c}\right]^\alpha \left[\frac{c_0+c}{(c_0-a)^\alpha} + \frac{c_0-c}{(c_0+a)^\alpha}\right];$$

$$u_m = \sigma_s/\rho_0 c_0 + (\sigma_m - \sigma_s)/\rho_0 c.$$

It follows from (6) and (8) that the stress $\sigma(0, t)$ and the mass flow rate $u(0, t)$ are related by the linear relationship

$$\sigma = \sigma_m + \rho_0 c_0 (K/L)(u - u_m)$$

on the boundary of the material in the elastic unloading domain.

By using the characteristic relationships (2), the boundary conditions (6) and (8) at the point P' on the axis t and the boundary conditions at the point P" on the unloading line OM, we obtain a solution of the problem

$$\begin{aligned}\sigma(h, t) &= \sigma_m\left\{1 - \frac{c_0+c}{2c}\left[\frac{a-c}{c_0-a}\left(\frac{c_0}{c}\frac{t}{T} - \frac{h}{cT}\right)\right]^\alpha + \frac{c_0-c}{2c}\left[\frac{a-c}{c_0+a}\left(\frac{c_0}{c}\frac{t}{T} + \frac{h}{cT}\right)\right]^\alpha\right\},\\ u(h, t) &= u_m - \frac{\sigma_m}{\rho_0 c_0}\left\{\frac{c_0+c}{2c}\left[\frac{a-c}{c_0-a}\left(\frac{c_0}{c}\frac{t}{T} - \frac{h}{cT}\right)\right]^\alpha + \frac{c_0-c}{2c}\left[\frac{a-c}{c_0+a}\left(\frac{c_0}{c}\frac{t}{T} + \frac{h}{cT}\right)\right]^\alpha\right\}\end{aligned} \tag{9}$$

for an arbitrary point P in domain IV (Fig. 2). Differentiating σ with respect to t and taking account of the condition $at - h \geqslant 0$, which is satisfied in domain IV, we obtain that $\partial\sigma/\partial t < 0$ in this domain, i.e., unloading of the material actually occurs.

The stresses in the plastic load wave reach a maximum on the line OM'. The distribution of these maximal stresses over the coordinate h is determined by the expression

$$\sigma_0(h) = \sigma_m\left[1 - \left(\frac{a-c}{a}\frac{h}{cT}\right)^\alpha\right]. \tag{10}$$

The line $H = at$ intersects the head C^+ characteristic AM on which the stress equals σ_s, at a point with the coordinates

$$\frac{t_*}{T} = \frac{c}{a-c}\left(-\frac{t_A}{T}\right), \quad \frac{h_*}{cT} = \frac{a}{a-c}\left(-\frac{t_A}{T}\right),$$

where

$$\left(-\frac{t_A}{T}\right)=\left(1-\frac{\sigma_s}{\sigma_m}\right)^{1/\alpha}.$$

Furthermore, depending on the amplitude of the boundary pulse σ_m, different flow modes can be realized. If the amplitude satisfies the inequality

$$\frac{\sigma_m}{\sigma_s}<1+\frac{2c\left(\frac{c_0-a}{c_0+a}\right)^{\alpha}}{(c_0+c)\left[1-\left(\frac{c_0-a}{c_0+a}\right)^{\alpha}\right]}=A, \tag{11}$$

then the C^- characteristic emerging from the point (t_*, h_*) will intersect the t axis at a time when the action of the boundary pulse has still not ceased.

The length of the plastic deformation zone is determined by the coordinate h_*. The duration of the purely elastic stress pulse passing into the material located farther than the section h_* is the same as for the initial boundary pulse, and only the apex of the stress pulse above σ_s is cut off.

If the amplitude of the boundary stress pulse satisfies the double inequality

$$A<\frac{\sigma_m}{\sigma_s}<1+\frac{2c}{(c_0-c)\left[1-\left(\frac{c_0-a}{c_0+a}\right)^{\alpha}\right]}=B, \tag{12}$$

where A is determined in (11), then a flow mode will be realized, for which the C^- characteristics emerging from the point (t_*, h_*) will intersect the t axis at a time when the action of the boundary pulse has already ceased. The C^+ characteristic emerging from the t axis at the time the action of the boundary pulse ceases will intersect the head C^+ characteristic AM of the plastic wave at a later time than the unloading line, hence the plastic flow will terminate at the point (t_*, h_*) as before. The length of the plastic deformation zone is determined by the coordinate h_*. However, the duration of the purely elastic pulse passing into the material to the right of h_* increases as compared to the duration of the boundary pulse.

Results of a computation of the problem whose solution by using a graphoanalytical method is presented in [1], are presented in Fig. 2. The parameters of the problem have the following values: $\alpha = 2$, $c/c_0 = 0.25$ $a/c_0 = 0.359$ (K = 1), $\sigma_m/\sigma_s = 2$. The boundary pulse is the parabola shown in the left upper part of Fig. 2. In this case the pulse amplitude satisfies the condition $\sigma_m/\sigma_s > B$, where B is determined in (12) so that the C^+ characteristic KL will intersect the line OM' at the point L to which arrives the C^+-characteristic of the plastic loading wave carrying a greater value of the stress than the yield point σ_s. Hence, the plastic flow has not yet terminated on the rectilinear section of the unloading line OL.

There are no stresses in the domain V on the boundary of the material above the point K, and the boundary velocity on the section KN is determined as a function of the time from the following relationship

$$\rho_0c_0u(0,t)=\frac{c_0-c}{c}\left\{\sigma_m\left[1-\left(\frac{c_0}{c}\frac{a-c}{c_0+a}\frac{t}{T}\right)^{\alpha}\right]-\sigma_s\right\}.$$

We hence obtain the solution for the domain V

$$\sigma(h,t)=\sigma_m\frac{c_0-c}{2c}\left(\frac{a-c}{c_0+a}\right)^{\alpha}\left[\left(\frac{c_0t+h}{cT}\right)^{\alpha}-\left(\frac{c_0t-h}{cT}\right)^{\alpha}\right],\quad \rho_0c_0u(h,t)=$$
$$=\frac{c_0-c}{c}(\sigma_m-\sigma_s)-\sigma_m\frac{c_0-c}{2c}\left(\frac{a-c}{c_0+a}\right)^{\alpha}\left[\left(\frac{c_0t+h}{cT}\right)^{\alpha}+\left(\frac{c_0t-h}{cT}\right)^{\alpha}\right] \tag{13}$$

and the domain VII

$$\rho_0c_0u(h,t)=\sigma(h,t)=\frac{c_0-c}{2c}(\sigma_m-\sigma_s)-\sigma_m\frac{c_0-c}{2c}\left(\frac{a-c}{c_0+a}\frac{c_0t-h}{cT}\right)^{\alpha}. \tag{14}$$

The equation of the next, generally, curvillinear section LMR of the unloading line can be obtained by expressing the C^+-invariant (2) in terms of values of the flow variables along a part of the boundary KN and in terms of values in the plastic loading wave, which yields the relationship

$$\frac{c_0+c}{2c}\left(\frac{h-ct}{cT}\right)^\alpha-\frac{c_0-c}{2c}\left(\frac{a-c}{c_0+a}\frac{c_0t-h}{cT}\right)^\alpha=1. \tag{15}$$

We obtain for the cases $\alpha = 1$ and 2, respectively

$$\frac{h}{cT}=\frac{2c(c_0+a)}{c_0^2+2ac_0+c^2}+\frac{2c_0c^2+a\left(c_0^2+c^2\right)t}{c\left(c_0^2+2ac_0+c^2\right)T}$$

or $h/cT = b + \lambda/c\cdot t/T$, where $c < \lambda < a$,

$$\frac{h}{cT}=\frac{\left(1-\delta\frac{c_0}{c}\gamma^2\right)\frac{t}{T}+\sqrt{\delta\gamma^2\left(\frac{c_0-c}{c}\right)^2\frac{t^2}{T^2}+(1-\sigma\gamma^2)\frac{2c}{c_0+c}}}{1-\sigma\gamma^2},$$

where $\gamma = (a-c)/(c_0+a)$; $\delta = (c_0-c)/(c_0+c)$.

A parametric representation of the curve LMR can be obtained for arbitrary α. Let us put $z = (h - cT)/cT$ and $\xi = (c_0t - h)/c_0T$. Then the relation between these parameters is determined by (15). Two forms of the parametric representation of the curve LMR appear as follows

$$\frac{t}{T}=\frac{c}{c_0-c}\left[z+\frac{c_0+a}{a-c}\left(\frac{c_0+c}{c_0-c}z^\alpha-\frac{2c}{c_0-c}\right)^{1/\alpha}\right],\quad \frac{h}{cT}=\frac{t}{T}+z,$$

where z varies between the value at the point L ($z_L = (h_L - ct_L)/cT$) and the value at the point M or A ($z_A = (1-\sigma_s/\sigma_m)^{1/\alpha}$):

$$\frac{t}{T}=\frac{c_0}{c_0-c}\left\{\xi+\frac{c}{c_0}\left[\frac{2c}{c_0+c}+\frac{c_0-c}{c_0+c}\left(\frac{a-c}{c_0+a}\frac{c_0}{c}\xi\right)^\alpha\right]^{1/\alpha}\right\},$$
$$\frac{h}{cT}=\frac{c_0}{c}\left(\frac{t}{T}-\xi\right), \tag{16}$$

where ξ varies in the corresponding interval from the value at the point L or K to the value at the point M.

The solution of the problem in the domain VI can be represented in the form

$$\frac{c_0t+h}{cT}=\frac{c}{c_0-c}\left[2z+\frac{c_0+c}{c_0}\frac{c_0+a}{a-c}\left(\frac{c_0+c}{c_0-c}z^\alpha-\frac{2c}{c_0-c}\right)^{1/\alpha}\right],$$
$$\sigma=\frac{c_0-c}{2c}\sigma_m\left[z^\alpha-\left(\frac{a-c}{c_0+a}\frac{c_0t-h}{cT}\right)^\alpha\right],$$
$$\rho_0c_0u=\frac{c_0-c}{2c}\sigma_m\left[2\frac{\sigma_m-\sigma_s}{\sigma_m}-z^\alpha-\left(\frac{a-c}{c_0+a}\frac{c_0t-h}{cT}\right)^\alpha\right].$$

If the second and third equations are solved for z and substituted into the first equation, then two relationships are obtained that determine σ and u as implicit functions of h and t.

The solution at any point of the domain VIII is determined easily since the values of the stress and the mass flow rate are known on the boundary NS as a function of the time: the stress is $\sigma = 0$ and the mass flow rate is determined in the parametric form

$$\rho_0c_0u=\frac{c_0-c}{c}\sigma_m\left(\frac{\sigma_m-\sigma_s}{\sigma_m}-z^\alpha\right)=w(t)\,\rho_0c_0,$$
$$\frac{t}{T}=\frac{c_0}{c_0-c}\left[2z+\frac{c_0+c}{c_0}\frac{c_0+a}{a-c}\left(\frac{c_0+c}{c_0-c}z^\alpha-\frac{2c}{c_0-c}\right)^{1/\alpha}\right]$$

or in implicit form if the first equation is solved for z and substituted into the second. For an arbitrary point (h, t) we have

$$\sigma(h,\ t) - \rho_0 c_0 u(h,\ t) = -\rho_0 c_0 w(t + h/c_0),$$
$$\sigma(h,\ t) + \rho_0 c_0 u(h,\ t) = \rho_0 c_0 w(t - h/c_0).$$

In domain IX we have $\sigma(h, t) - \rho_0 c_0 u(h, t) = 0$, $\sigma(h, t) + \rho_0 c_0 u(h, t) = \rho_0 c_0 w(t - h/c_0)$. formulas defining σ and u as implicit functions of h and t can be obtained for these domains.

The rest state holds in domain X.

Dependences of the stress on the time in three sections, h = cT, h = 2cT, and $h = h_M$, are presented in the lower part of Fig. 2. It is seen from a comparison between these graphs and those presented in [1] that the graphoanalytic method does not expose the characteristic singularity of the solution in the unloading domain in this case. For instance, the nonmonotonic nature of the stress changes on the section ABCD and also the explicit breaks on the appropriate characteristics do not appear in the section h = cT in the graphoanalytical approach. The third graph shows that the graphoanalytical method gave an incorrect direction for the convexity on the section ABC for the purely elastic pulse passing into the material after the plastic flow has terminated. A complete unloading wave, starting at the point L on the characteristic KL and terminating at the point R on the characteristic NR, and corresponding to a change in the parameter ξ in (16) from the value ξ_K to the value ξ_N is shown on the h–t diagram in Fig. 2 by the dashed curve LMR. This curve is independent of the specific value of the quantity σ_m/σ_s. As the value of σ_m/σ_s increases, the point M will advance along this curve towards the point R and will reach R for the value

$$\frac{\sigma_m}{\sigma_s} = \left(\frac{c_0 + c}{c_0 - c}\right)^2 \frac{1 - \dfrac{c_0 - c}{c_0 + c}\left(\dfrac{c_0 - a}{c_0 + a}\right)^\alpha}{1 - \left(\dfrac{c_0 - a}{c_0 + a}\right)^\alpha}.$$

As the amplitude increases further, a second curvilinear section of the unloading wave appears which is determined by the boundary conditions on the part of the boundary NS, etc. to infinity. This results from the momentum conservation law. To conserve the total momentum as the amplitude of the elastic pulse passing into the undeformed plastic material diminishes to σ_s, its duration should increase in an appropriate manner.

However, the following two circumstances must be remembered for specific computations. Firstly, the Prandtl scheme becomes inapplicable for very large pressures. For instance, in the case of uniaxial compression deformation there results from the Prandtl scheme: a zero specific volume, i.e., infinite material density, is achieved for a finite compression stress, which is physically absurd. Secondly, reverse plastic flow can start during unloading. This means that a plastic unloading wave line, behind which will be a domain of reverse plastic flow, should emerge from the t axis between 0 and t_K on the h–t-diagram at a certain time. In this case, new boundary conditions on this line will influence the solution of the problem so that the solution elucidated above will become inapplicable.

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